MID-SEMESTER EXAM | ISI BANGALORE | SEPTEMBER 2022

DIFFERENTIAL GEOMETRY I | 100 POINTS

Notes.

(a) You may freely use any result proved in class or in the textbook unless you have been asked to prove the same. Use your judgement. All other steps must be justified.

(b) There are a total of **108** points in the paper. You will be awarded a maximum of **100**.

(c) $\mathbb{R}$ = real numbers.

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1. [6+6+6=18 points] Give brief answers:
   (i) Find the regular and critical values of $f(x,y) = y^2 - x^2 - x^3$.
   (ii) Let S be the cone in $\mathbb{R}^3$ given by $x^2 + y^2 = z^2$, $z > 0$. Pick an orientation on S and identify the image of the Gauss map induced by it.
   (iii) Find the geodesic on the cylinder $x^2 + y^2 = 4$ in $\mathbb{R}^3$ through the point $p = (2, 0)$, having tangent vector $v = (0, 1, 1)$ at p.

2. [6+6+6=18 points] Consider the map $\sigma(u,v) = (\frac{1}{u}, u + \sin v, v + \cos(\pi u), e^{uv}) \colon U \to \mathbb{R}^4$ where $U \subset \mathbb{R}^2$ is given by $u \neq 0$.
   (i) Verify that the image S of σ is a 2-manifold in $\mathbb{R}^4$.
   (ii) At the point $p = (1, 1, -1, 1) \in S$, identify the tangent plane T_pS as a subspace of $\mathbb{R}^4$.
   (iii) At p as in (ii), pick a nonzero tangent vector v in $T_p(S)$ and find a parametrised curve $\alpha(t)$ on S such that $\alpha(0) = p$ and $\dot{\alpha}(0) = v$.

3. [18 points] Let S be the surface in $\mathbb{R}^3$ obtained by identifying two opposite sides of a rectangular band after twisting it through a rotation of 3π along a central line. More precisely, S is parametrised by
$$\sigma(t,\theta) = ((1 - t\sin(\tfrac{3\theta}{2}))\cos\theta,\ (1 - t\sin(\tfrac{3\theta}{2}))\sin\theta,\ t\cos(3\theta/2)),$$
and is covered by the 2 open sets $\sigma(U_i)$ where
$$U_1 = \{(t,\theta) \in (\tfrac{-1}{2}, \tfrac{1}{2}) \times (0, 2\pi)\}, \qquad U_2 = \{(t,\theta) \in (\tfrac{-1}{2}, \tfrac{1}{2}) \times (-\pi, \pi)\}.$$
Prove that S is non-orientable. (*Hint*: Compute the unit normals along the central line $t = 0$.)

4. [18 points] Let S be a k-manifold in $\mathbb{R}^n$. Let $\mathbf{X}$ be a smooth *tangent* vector field on S. Let $p \in S$. Prove that there exists an open interval I containing $0 \in \mathbb{R}$ and a parametrised curve $\alpha \colon I \to S$ such that $\alpha(0) = p$ and $\dot{\alpha}(t) = \mathbf{X}(\alpha(t))$ for all $t \in I$.

5. [18 points] Consider the regular curve $\alpha(t) = (t, t^2, t^3)$. Compute its Frenet frame and the curvature and torsion when $t = 1$.

6. [18 points] Let I be an open interval around $0 \in \mathbb{R}$. Given $p \in \mathbb{R}^2$, a unit vector $u \in \mathbb{R}^2_p$ and a smooth function $k(s) \colon I \to \mathbb{R}$, prove that there is *at most one* unit-speed curve $\alpha(s)$ in $\mathbb{R}^2$ such that
   (i) $\alpha(0) = p$, $\dot{\alpha}(0) = u$;
   (ii) The signed curvature of $\alpha(s)$ is $k(s)$.